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OF MODULATIONAL INSTABILITIES

L.C. HIMMELL

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A general formalism is developed to describe the nonlinear evolution associated with the parametric decay of an intense, coherent electromagnetic wave into an electrostatic wave, its second harmonic, and scattered electromagnetic waves in a homogeneous plasma. The effects of pump depletion are neglected and it is assumed that all waves are coherent. Two classes of solutions are found. One class is explosively unstable, while the other consists of growing aperiodic oscillations. The evolution of modulational modes is discussed.

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CONTENTS

ABSTRACT

A general formalism is developed to describe the nonlinear evolution associated with the parametric decay of an intense, coherent electromagnetic wave into an electrostatic wave, its second harmonic, and scattered electromagnetic waves in a homogeneous plasma. The effects of pump depletion are neglected and it is assumed that all waves are coherent. Two classes of solutions are found. One class is explosively unstable, while the other consists of growing aperiodic oscillations. The evolution of modulational modes is discussed.

I. INTRODUCTION

The parametric decay of an intense, electromagnetic wave into electrostatic waves and scattered electromagnetic waves in fully ionized plasmas has been studied extensively.[1-13] These studies have wide applications to laser fusion experiments, interactions of pulsar radiation with a plasma, ionospheric modification experiments, etc. There is still considerable interest in the nonlinear evolutionary properties of these waves since they ultimately determine the importance of a particular process to an experiment. It is also well known that certain nonlinearities completely change the nature of a particular mode; e.g., they may lead to explosive instabilities.

In this study, we examine some nonlinear properties associated with the parametric decay of an intense, coherent, electromagnetic wave into an electrostatic wave, its second harmonic and scattered electromagnetic waves in a homogeneous plasma. For the case of scattering off ion acoustic waves, the mode coupling processes of many harmonics must generally be considered. Nonetheless, there are cases in which a modulational mode and only one harmonic can exist and it is situations of this kind that we consider.

We find that general conditions exist for which explosive instabilities result, and that this phenomenon may occur after about a linear e-folding time.

II. GENERAL FORMALISM

Consider the propagation of a large-amplitude, linearly polarized, coherent electromagnetic pump wave,

$$\vec{E}_o = 2\, E_o\, \hat{e}_o \cos(\vec{k}_o \cdot \vec{x} - \omega_o t) \quad , \tag{1}$$

in a homogeneous plasma. In the absence of damping, the pair (ω_o, k_o) satisfies the usual dispersion relation,

$$\omega_o{}^2 = k_o{}^2 c^2 + \omega_p{}^2 \quad , \tag{2}$$

provided that the electron quiver velocity $\vec{v}_o = (-e\vec{E}_o/m\omega_o)$ is non-relativistic.[14-15]

We consider the case in which the pump wave excites a low-frequency modulational mode (ω, k) and its harmonic $(2\omega, 2k)$, as well as the two lowest-order high-frequency sidebands, $(\omega_\pm = \omega \pm \omega_o,\ \vec{k}_\pm = \vec{k} \pm k_o)$, $(\omega_{2\pm} = 2\omega \pm \omega_o,\ \vec{k}_{2\pm} = 2\vec{k} \pm \vec{k}_o)$, associated with each wave. The effects of pump depletion are neglected and it is assumed that all waves are coherent. It is shown below that, under these conditions, if the two modes satisfy certain general properties, an explosive instability results and the waves become unbounded in finite time.

First, we calculate the perturbed current densities, since they are the sources that determine the nonlinear evolution of the waves. In first order, they are as follows:

$$\vec{j}_\pm{}^{(1)} = \sigma_\pm \vec{E}_\pm \mp \frac{e}{m} P_k \chi_e(k,\omega) \frac{k\vec{E}_o}{4\pi\omega_o} \tag{3a}$$

$$j_k{}^{(1)} = \frac{\omega}{4\pi i} \left[P_k \chi_e(k,\omega) + E_k \chi_i(k,\omega)\right] \quad , \tag{3b}$$

where $\sigma_\pm = i\omega_p{}^2/4\pi\omega_\pm$ is the linear plasma conductivity at frequencies $\omega_\pm$, $\chi(k,\omega) = (\omega_p{}^2)/(n_o k^2) \int d\vec{v}\vec{k}\cdot\nabla_v f_o/(\omega-\vec{k}\cdot\vec{v})$ is the linear susceptibility, and

$$\vec{P}_k = \vec{E}_k - \frac{i\vec{k}}{\omega_o}(\vec{E}_+ + \vec{E}_-)\cdot\vec{v}_o \tag{4}$$

is the linear field, including the ponderomotive contribution, governing the motion of the electrons at frequency ω.[1]

In second order, the Vlasov equations for the perturbed electron density distributions take the form:

$$-i(\omega-\vec{k}\cdot\vec{v})f_k^{(2)} - \frac{e}{m}\left[\vec{P}_k^{(2)}\cdot\vec{\nabla}_v f_o + \vec{P}_{2k}\cdot\nabla_v f_{-k} + \vec{P}_{-k}\cdot\vec{\nabla}_v f_{2k}\right] = 0 \ , \tag{5a}$$

$$-2i(\omega-\vec{k}\cdot\vec{v})f_{2k}^{(2)} - \frac{e}{m}\left[\vec{P}_{2k}^{(2)}\cdot\vec{\nabla}_v f_o + \vec{P}_k\cdot\vec{\nabla}_v f_k\right] = 0 \ , \tag{5b}$$

$$\begin{aligned}-i(\omega_\pm-\vec{k}_\pm\cdot\vec{v})f_\pm^{(2)} - \frac{e}{m}\Big[&\vec{E}_o\cdot\vec{\nabla}_v f_k^{(2)} + \vec{E}_k\cdot\vec{\nabla}_v f_{\pm k_o} \\ &+ \vec{E}_{2k}\cdot\vec{\nabla}_v(f_\mp^{(1)*}) + \vec{E}_{2\pm}\cdot\vec{\nabla}_v f_{-k}^{(1)} + \vec{E}_{-k}\cdot\vec{\nabla}_v f_{2\pm} + \vec{E}_\mp{}^*\cdot\nabla_v f_{2k}^{(1)}\Big] = 0 \ ,\end{aligned} \tag{5c}$$

$$\begin{aligned}-i(\omega_{2\pm} - \vec{k}_{2\pm}\cdot\vec{v})f_{2\pm}^{(2)} - \frac{e}{m}\Big[&\vec{E}_o\cdot\vec{\nabla}_v f_{2k}^{(2)} + \vec{E}_{2k}\cdot\vec{\nabla}_v f_{\pm k_o}^{(1)} \\ &+ \vec{E}_k\cdot\vec{\nabla}_v f_\pm^{(1)} + \vec{E}_\pm\cdot\nabla_v f_k^{(1)}\Big] = 0 \ ,\end{aligned} \tag{5d}$$

where

$$\vec{P}_k^{(2)} = \frac{e}{m\omega_o{}^2}\, i\vec{k}\left[\vec{E}_-^*\cdot\vec{E}_{2-} + \vec{E}_+{}^*\cdot\vec{E}_{2+}\right] \tag{6a}$$

and

$$\vec{P}_{2k}^{(2)} = \frac{e}{m\omega_o{}^2}\, i\vec{k}\,\vec{E}_+\cdot\vec{E}_- \tag{6b}$$

are the fields that determine the second-order electron dynamics.[16] The second-order, perturbed current densities are due to the electron motion. They can be reduced to the following:

$$j_k^{(2)} = -\frac{e\omega}{k} n_k^{(2)} \tag{7a}$$

$$j_{2k}^{(2)} = -\frac{e\omega}{k} n_{2k}^{(2)} \tag{7b}$$

$$\vec{j}_{\pm}^{(2)} = \pm \frac{e^2 i}{m\omega_o}\left[\vec{E}_o\, n_k^{(2)} + \vec{E}_{2\pm} n_k^{(1)*} + \vec{E}_{\mp}^{\,*}\, n_{2k}^{(1)}\right] \tag{7c}$$

$$j_{2\pm}^{(2)} = \pm \frac{e^2 i}{m\omega_o}\left[\vec{E}_o n_{2k}^{(2)} + \vec{E}_{\pm} n_k^{(1)}\right] \tag{7d}$$

where

$$n_k^{(1)} = \frac{ik}{4\pi e}\, \chi_e\,(k,\omega)P_k\ , \tag{8a}$$

$$n_k^{(2)} = \frac{ik}{4\pi e}\left[\chi_e\,(k,\omega)P_k^{(2)} + \frac{iek}{4m}\left[\frac{\partial^2}{\partial\omega^2}\chi_e(k,\omega)\right]P_{2k}P_k^{*}\right] \tag{8b}$$

$$n_{2k}^{(2)} = \frac{ik}{8\pi e}\left[\chi_e\,(k,\omega)P_{2k}^{(2)} - \frac{iek}{2m}\left[\frac{\partial^2}{\partial\omega^2}\chi_e(k,\omega)\right]P_k^{2}\right] \tag{8c}$$

are the perturbed densities corresponding to (ω,k) and $(2\omega, 2k)$ in lowest order.*

The Fourier-transformed wave equations for E_k and $\vec{E}_{\pm}$ take the form:

$$-\omega^2(1+\chi_i(k,\omega))E_k = \omega^2\chi_e(k,\omega)P_k + 4\pi i\omega j_k^{(2)} \tag{9}$$

$$\underset{\sim}{M}_{\pm}\cdot\vec{E}_{\pm} = -\frac{iek}{m}\vec{E}_o\chi_e(k,\omega)P_k + 4\pi i\omega_{\pm}\vec{j}_{\pm}^{(2)} \tag{10}$$

where

$$\underset{\sim}{M}_{\pm} = [(k_{\pm}c)^2 - \omega_{\pm}^{\,2}\epsilon_{\pm}]\underset{\sim}{I} - \vec{k}_{\pm}\vec{k}_{\pm}c^2\ , \tag{11}$$

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*See Appendix.

$\varepsilon_\pm$ is the linear dielectric constant corresponding to $(\omega_\pm, k_\pm)$ and $\underset{\sim}{I}$ denotes the unit dyadic. The equations for E_{2k} and $\vec{E}_{2\pm}$ are similar.

Next, the term involving P_k is eliminated from Eqs.(9-10) by using Eq.(4). Solving for P_k, we obtain

$$\Delta(\omega,k)P_k = -\frac{4\pi i}{\omega}\, j_k^{(2)} + \frac{k\vec{v}_o}{\omega_o}\cdot\left[\omega_+\underset{\sim}{M}_+^{-1}\cdot j_+^{(2)} + \omega_-\underset{\sim}{M}_-^{-1}\cdot j_-^{(2)}\right]4\pi(1+\chi_i(k,\omega)) \; , \tag{12}$$

where

$$\Delta(\omega,k) = 1+\chi_i(k,\omega) + \chi_e(k,\omega) - k\vec{v}_o\cdot(M_+^{-1} + M_-^{-1})\cdot k\vec{v}_o\chi_e(k,\omega)(1+\chi_i(k,\omega)) \tag{13}$$

is the modified linear dielectric constant in the presence of $\vec{E}_o$, and

$$\underset{\sim}{M}_\pm^{-1} = \left[\underset{\sim}{I} - \frac{\vec{k}_\pm\vec{k}_\pm c^2}{\omega_\pm{}^2\varepsilon_\pm}\right] / D_\pm \; , \tag{14a}$$

with

$$D_\pm = (k_\pm c)^2 - \omega_\pm{}^2\varepsilon_\pm \; . \tag{14b}$$

Using this expression for P_k, Eqs.(9-10) reduce to the following:

$$\Delta(\omega,k)E_k = J(\omega,k)/(1+\chi_i(k,\omega)) \quad , \tag{15a}$$

$$\Delta(\omega,k)\vec{E}_\pm = -\, iM_\pm^{-1}\cdot\vec{v}_o \;\; k\omega_o\, J(\omega,k) \quad , \tag{15b}$$

where

$$J(\omega,k) = \chi_e(k,\omega)4\pi i\left[\frac{j_k^{(2)}}{\omega} + \frac{ik\vec{v}_o}{\omega_o}\cdot\left(\omega_+M_+^{-1}\cdot j_+^{(2)} + \omega_-M_-^{-1}\cdot j_-^{(2)}\right)(1+\chi_i(k,\omega))\right] \tag{16}$$

with similar expressions for E_{2k} and $\vec{E}_{2\pm}$. Since

$$\Delta(\omega,k) = 0 \tag{17}$$

is the linear dispersion relation, terms proportional to $\Delta(\omega,k)$ have been neglected on the right hand side of Eqs.(15a-15b).

Due to linear instability and nonlinear coupling, the waves $E_k, E_{2k}, \ldots$ etc. will vary on a time scale roughly of the order of a linear e-folding time. The evolutionary equations for the wave amplitudes, obtained in the usual way, are as follows:†

$$\left(\frac{\partial}{\partial t} - \gamma(k)\right)\begin{bmatrix} E_k \\ \vec{E}_\pm \end{bmatrix} = \frac{-i}{\left.\frac{\partial}{\partial \omega}\Delta\right|_{\omega_R(k)}} \, exp[-i(\omega_R(2k)-2\omega_R(k))t]$$

$$\times \frac{J(\omega,k)}{(1+\chi_i(\omega,k))}\begin{bmatrix} 1 \\ -\underset{\sim}{M}_\pm^{-1}\cdot\vec{v}_o \; ik\omega_o(1+\chi_i(k,\omega)) \end{bmatrix} \tag{18a}$$

$$\left(\frac{\partial}{\partial t} - \gamma(2k)\right)\begin{bmatrix} E_{2k} \\ \vec{E}_{2\pm} \end{bmatrix} = \frac{-i}{\left.\frac{\partial}{\partial \omega}\Delta\right|_{\omega_R(2k)}} \, exp[-i(2\omega_R(k)-\omega_R(2k))t]$$

$$\times \frac{J(2\omega,2k)}{(1+\chi_i(2k,2\omega))}\begin{bmatrix} 1 \\ -\underset{\sim}{M}_{2\pm}^{-1}\cdot\vec{v}_o i2k\omega_o(1+\chi_i(2k,2\omega)) \end{bmatrix} \tag{18b}$$

where $\omega_R = \text{Real}(\omega)$ and the fields are assumed to be independent of position.[17]

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†Note that Eqs.(18) must be modified for stimulated Brillouin scattering, since in that case,

$$\Delta(\omega + i\frac{\partial}{\partial t},k) = \Delta(\omega,k) + i\frac{\partial}{\partial t}\frac{\partial}{\partial \omega}\Delta(\omega,k) + \cdots$$

does not converge for $(\partial/\partial t)P_K << \omega P_k$. This is because $M_-^{-1} \propto 1/(\omega-\Delta\omega + i\Gamma_-)$, and $\omega \simeq \Delta\omega \simeq kC_s$ for ion acoustic modes, so that M^{-1} cannot be expanded in a Taylor series about ω.

Using a linearized version of Eq.(10), $J(\omega,k)$ and $J(2\omega,2k)$ can be reduced to a term proportional to $P_{2k}P_k^*$ and P_k^2 respectively, where the proportionality constant is in general complex. Eqs.(18a), (18b) can then be transformed, using Eq.(4), into a pair of equations involving P_k and P_{2k} only, which take the form

$$\left(\frac{\partial}{\partial t} - \gamma\right)P_k = c_1 P_{2k} P_k^* \quad , \tag{19}$$

$$\left(\frac{\partial}{\partial t} - \gamma_2\right)P_{2k} = c_2 P_k^2 \quad , \tag{20}$$

respectively, where

$$c_1 = \frac{i}{\left.\left(\frac{\partial}{\partial \omega}\Delta\right)\right|_{\omega_R(k)}} \frac{\tilde{J}}{\chi_e(k,\omega)} \quad , \tag{21a}$$

$$c_2 = \frac{i}{\left.\left(\frac{\partial}{\partial \omega}\Delta\right)\right|_{\omega_R(2k)}} \frac{\tilde{J}_2}{\chi_e(2k,2\omega)} \quad , \tag{21b}$$

$$\tilde{J} = J(\omega,k)/P_{2k}P_k^* \quad , \tag{21c}$$

$$\tilde{J}_2 = J(2\omega,2k)/P_k^2 \quad , \tag{21d}$$

the small frequency mismatches have been neglected, and terms involving $k\vec{v}_o \cdot (M_+^{-1} + M_-^{-1}) \cdot k\vec{v}_o$ were eliminated by using the linear dispersion relation, Eq.(17).

A solution to Eqs.(19),(20) is sufficient to determine the nonlinear evolution of all wave amplitudes, provided that appropriate initial conditions are given. [18] They apply to all parametric processes for which both $\Delta(\omega,k) \approx 0$ and $\Delta(2\omega,2k) \approx 0$.

III. ANALYSIS

We now examine the properties of Eqs.(19),(20). First, we transform them into a dimensionless polar form. Setting $\tau = \gamma t$, $A_k = [|c_1 c_2|]^{1/2} P_k/\gamma$, and $A_{2k} = c_1 P_{2k}/\gamma$, we obtain

$$\left(\frac{d}{d\tau} - 1\right)A_k = A_{2k}A_k^* \quad , \tag{22}$$

$$\left(\frac{d}{d\tau} - \tilde{\gamma}\right)A_{2k} = e^{i\psi}A_k^2 \quad , \tag{23}$$

where $\psi = arg(c_1 c_2)$ and $\tilde{\gamma} = \gamma(2k)/\gamma(k)$. In the polar representation, $A_k = a_k e^{i\phi k}$, $A_{2k} = a_{2k}e^{i\phi 2k}$, Eqs.(22),(23) take the form

$$\frac{d}{d\tau} a_k = a_k(1 + a_{2k}\cos\phi) \quad , \tag{24}$$

$$\frac{d}{d\tau} a_{2k} = \tilde{\gamma}a_{2k} + a_k^2\cos(\phi-\psi) \quad , \tag{25}$$

$$\frac{d}{d\tau}\phi = \frac{a_k^2}{a_{2k}}\sin(\psi-\phi) - 2a_{2k}\sin\phi \quad , \tag{26}$$

where $\phi = \phi_{2k} - 2\phi_k$. Finally, setting $x = a_k^2 e^{-2\tau}$, $y = a_{2k}e^{-\tilde{\gamma}\tau}$, Eqs.(24) through (26) reduce to the following:

$$\frac{d}{d\tau} x = 2xy\cos\phi e^{\tilde{\gamma}\tau} \quad , \tag{27}$$

$$\frac{dy}{d\tau} = x\cos(\phi-\psi)e^{(2-\tilde{\gamma})\tau} \quad , \tag{28}$$

$$y\frac{d\phi}{d\tau} = e^{(2-\tilde{\gamma})\tau}[x\sin(\psi-\phi) - 2y^2 e^{-2(1-\tilde{\gamma})\tau}\sin\phi] \quad , \tag{29}$$

which imply

$$\frac{d}{d\tau}\, xy\, \sin\phi = x^2 e^{(2-\tilde{\gamma})\tau} \sin\psi \quad . \tag{30}$$

Eqs.(27)-(29) are now in a convenient form to be analyzed. There are two general cases of solutions depending upon the sign of $\cos\psi$.

(i) $\underline{\cos\psi > 0}$:

We divide ϕ space into four regions. For ψ in Quadrant I , let $c = \{\phi : 0 < \phi < \psi\}$ and $c^- = \{\phi : \psi < \phi < \pi\}$, while for ψ in Quadrant IV, set $c = \{\phi : \psi < \phi < 0\}$ and $c^- = \{\phi : 0 < \phi < \psi + \pi\}$. Let $c^+ = c^- + \pi$ and $c^* = c + \pi$. According to Eq.(29), $sgn(d\phi/d\tau) = \pm 1$ for $\phi \in \left\{\begin{matrix} c^+ \\ c^- \end{matrix}\right.$ which implies that, if $\phi \in c^+ \cup c^-$, then ϕ changes monotonically until it enters c , where it is trapped.

For $\phi \in c$, according to Eqs.(27),(28), x and y increase. A lower bound for (x,y) is obtained from a solution to

$$\frac{dx}{d\tau} = 2xy \cos\psi \ , \tag{31}$$

$$\frac{dy}{d\tau} = x \cos\psi \ , \tag{32}$$

provided that $\tilde{\gamma} \le 2$. This corresponds to the modulational modes discussed below. The general solution to Eqs.(31),(32) is as follows:

$$y(\tau) = \begin{cases} M^{1/2} \tan[\lambda + \tan^{-1}(y'/M^{1/2})] & \text{for } M > 0 \ , \\ \left[\dfrac{y' \cosh\lambda - |M|^{1/2} \sinh\lambda}{|M|^{1/2} \cosh\lambda - y' \sinh\lambda}\right] |M|^{1/2} & \text{for } M < 0 \ , \end{cases} \tag{33}$$

where

$$M = x - y^2 \tag{34}$$

is a constant of the motion, $\lambda = |M|^{1/2}(\tau - \tau')\cos\psi$, and the prime refers to to an initial value corresponding to the time at which ϕ enters c .

According to Eqs.(33),(34), x and y become unbounded when

$$\tau - \tau' = \tau_\infty = \begin{cases} [\pi/2 - \tan^{-1}(y'/M^{1/2})]/M^{1/2}\cos\psi & \text{for } M > 0\ , \\ -\dfrac{1}{2|M|^{1/2}\cos\psi}\, \ln\left[\dfrac{y' - |M|^{1/2}}{y' + |M|^{1/2}}\right] & \text{for } M < 0\ , \end{cases} \tag{35}$$

thus providing a conservative estimate of the time at which x and y become infinite. For $\phi \in c^*$, $d\phi/d\tau$ can have either sign.

(ii) <u>$\cos\psi < 0$</u>:

In this case, let $c = \{\phi: 0 < \phi < \psi\}$, $c^- = \{\phi: \psi < \phi < \pi\}$, $c^* = c + \pi$, and $c^+ = c^- + \pi$ for ψ in Quadrant II, and let $c = \{\phi: \psi < \phi < 0\}$, $c^- = \{\phi: 0 < \phi < \psi + \pi\}$, $c^* = c + \pi$ and $c^+ = c^- + \pi$ for ψ in Quadrant III. In general, for $\phi \in c^- \cup c^+$, ϕ changes monotonically as before until it enters c, where it is "trapped", and in general oscillates.

The pair (x,y), however, may become explosively unstable. Assume for definiteness that ψ is in Quadrant II. When $\psi - \pi/2 < \phi < \pi/2$, x and y both increase with time. Consider the region $\psi - \bar\phi < \phi < \bar\phi$ for some $\bar\phi$, such that $\psi - \pi/2 < \bar\phi < \pi/2$. A lower bound for $(x(\tau), y(\tau))$ is obtained from a solution to the equations

$$\frac{d}{d\tau} x = 2xy \cos\bar\phi\ , \tag{36}$$

$$\frac{d}{d\tau} y = x \cos\bar\phi\ , \tag{37}$$

which is described by Eqs.(33)-(35), if ψ is replaced by $\bar\phi$. The pair (x,y) therefore becomes explosively unstable if ϕ remains in this region for a time equal to τ_∞ as given by Eq.(35).

If $\phi(o) \in c^*$, $d\phi/d\tau$ can have either sign.

IV. SCATTERING OFF MODULATIONAL MODES

Modulational modes are characterized by very long wavelengths, $k \ll k_o$. They resonate with both Stokes and anti-Stokes waves and are such that $\vec{k}\cdot\vec{E}_o = 0$. In this case, $\omega \propto \vec{k}\cdot\vec{v}_g$, where $\vec{v}_g = (\vec{k}_o c^2/\omega_o)$ is the group velocity of the pump.

In the limit $k \ll k_o$, $\omega \ll kc$, the dielectric constant reduces to

$$\Delta(\omega,k) = 1 + \chi_i + \chi_e + 2\chi_e(1+\chi_i)\left(\frac{v_o^2\delta^2}{c^2}\right)\left[\frac{1}{(\omega-k\cdot v_g+i\Gamma)^2-\delta^2}\right] \tag{38}$$

where $\delta = (kc)^2/2\omega_o$ and Γ, the collisional damping rate of the free electromagnetic wave, has been added.

We consider the following case:[1,2]

1) $kv_i \ll \omega \ll kv_e$, $\omega \ll \omega_{p_i}$, $\delta^2 \ll (\omega-\vec{k}\cdot\vec{v}_g)^2$, $\omega^2 \gg \omega_i^{\,2}$

$\Gamma \ll \mathrm{Im}(\omega)$, $(k\lambda_D)^2 \ll 1$, $(\omega_i \equiv \omega_{p_i}/\sqrt{1+(k\lambda_D)^{-2}})$.

Modes exist with frequencies

$$\omega(k) = \frac{k\cdot v_g}{2} \pm \frac{1}{2}\sqrt{(k\cdot v_g)^2 - 4\sqrt{2}\,(v_o/c)\delta\omega_{p_i}} \tag{39}$$

which become unstable for $\omega^2 \le \sqrt{2}\,\frac{v_o}{c}\,\delta\omega_{p_i}$. Since $\omega \propto k$, $(2\omega,2k)$ is also a mode and $\gamma(2k) = 2\gamma(k)$. Therefore, the frequency mismatch $\omega(2k) - \omega(k)$ in Eqs.(18) is exactly equal to zero. In this regime, $\chi_e(k,\omega) \approx (k\lambda_D)^{-2}(1+i\sqrt{\pi}(\omega/kv_e))$, $\chi_i(k,\omega) \approx -(\omega_{p_i}^{\,2}/\omega^2)$, and we find that

$$\left.\frac{\partial}{\partial\omega}\Delta\right|_{\mathrm{Re}\,\omega(k)} = 2\left(\frac{\omega_{p_i}^{\,2}}{\omega^3}\right)[1 - (\delta/\omega_i)^2] + \left(i(k\lambda_D)^{-2}/\omega\right)\left[\sqrt{\pi}\,\frac{\omega}{kv_e} + 2\Gamma/\omega\right]. \tag{40}$$

Next, we calculate $J = J(\omega,k)$ and $J_2 = J(2\omega,2k)$. Assuming that $E_{2k} \sim E_k$, we find that $\vec{j}_\pm = \pm \frac{e^2 i E_0^*}{m\omega_0} n_k^{(2)}$, where terms of order $4\frac{\delta}{\omega}\left(\frac{\omega_i}{\omega}\right)^2 \ll 1$ are neglected. Also,

$$n_k^{(2)} = \frac{ik}{4\pi e} \chi_e P_k^{(2)}$$

if the second term in Eq.(8b), which is of the order

$$8\left(\frac{v_o}{v_e}\right)^2\left(\frac{\delta}{\omega}\right)^2\left(\frac{\omega_i}{\omega}\right)^4 ,$$

is neglected. Finally, comparing the two terms on the right-hand side of Eq.(16), we find that the second term is of order

$$\left(\frac{k}{k_o}\right)^2\left(\frac{v_o}{c}\right)^2 \ll 1$$

of the first and can therefore also be discarded. Since these results also apply to J_2, Eq.(16) can be reduced to the following:

$$J = \frac{iek}{8m\omega^2}\left(\frac{kv_o}{\omega_o}\right)^2 \chi_e^4 P_k^* P_{2k} \tag{41a}$$

$$J_2 = \frac{iek}{32m\omega^2}\left(\frac{kv_o}{\omega_o}\right)^2 \chi_e^4 P_k^2 \tag{41b}$$

where χ_e is to be evaluated at (ω,k). The coupling coefficients take the form

$$c_1 = -\frac{1}{8}\left(\frac{ek}{m\omega}\right)\left(\frac{kv_o}{\omega_o}\right)^2 \chi_e^3/[2|\chi_i|(1-(\delta/\omega_i)^2) + i\chi_e(\sqrt{\pi}\frac{\omega}{kv_e} + 2\Gamma/\omega)] \tag{42a}$$

$$c_2 = -\left(\frac{ek}{m\omega}\right)\left(\frac{kv_o}{\omega_o}\right)^2 \chi_e^3/[2|\chi_i|(1-4(\delta/\omega_i)^2) + i\chi_e(\sqrt{\pi}\frac{\omega}{kv_e} + \Gamma/\omega)] . \tag{42b}$$

which implies that

$$(2\omega_i^2/\omega^2)^2(1-(\delta/\omega_i)^2)(1-4(\delta/\omega_i)^2) > \pi(\omega/kv_e)^2 \tag{43}$$

for an explosive instability, where Γ/ω was neglected compared to ω/kv_e and, since $|\chi_i|/\chi_e \approx \omega_i^2/\omega^2$. Eq.(43) is a condition which can be readily satisfied.

The strength of the nonlinearity can be assessed by calculating the initial value $(x(o),y(o))$. According to the definition

$$x(o) = |c_1c_2 P_k/\gamma|^2$$

$$\approx \frac{1}{32}\left(\frac{ek}{m\omega}\right)^2\left(\frac{kv_o}{\omega_o}\right)^4 \frac{\chi_e^6}{\chi_i^2(1-(\delta/\omega_i)^2)(1-4(\delta/\omega_i)^2)} \left|\frac{P_k}{E_-}\right|^2 \left|\frac{E_-}{E_o}\right|^2 \frac{E_o^2}{\gamma^2} \tag{44}$$

where the imaginary parts of c_1 and c_2 have been neglected as a rough approximation. According to Eqs.(10), (14a),

$$|P_k/E_-|^2 \approx \left(\frac{kv_o}{2\omega}\chi_e\right)^{-2} .$$

Since $\gamma/\omega \ll 1$, $\omega^2 \approx \sqrt{2}\, \frac{v_o}{c}\, \delta\omega_{p_i}$ in Eq.(39) so that

$$(kv_o/\omega_o)^4\left(\frac{\omega_o}{\omega}\right)^2(k\lambda_D)^{-4} \approx (\omega/\omega_i)^4(\omega/\delta)^2 .$$

Eq.(44) then takes the form

$$x(o) \approx \frac{1}{8}\left(\frac{\omega}{\gamma}\right)^2\left(\frac{\omega}{\omega_i}\right)^8\left(\frac{\omega}{\delta}\right)^2 |E_-/E_o|^2/(1-(\delta/\omega_i)^2)(1-4(\delta/\omega_i)^2) \quad , \tag{45}$$

which indicates that the nonlinearity is strong. If, as an example, we take

$$(\delta/\omega_i)^2 = \frac{1}{8} \quad , \quad \omega/\gamma = 5 \quad , \quad (\omega/\omega_i)^2 = 5 \; , \quad m/M = \frac{1}{3600}$$

then, for an initial perturbation, $|E_-/E_0| \approx 2 \times 10^{-3}$, we find that $x(o) \approx 1$, $y(o) \approx \sqrt{|c_1/c_2|} \approx .3$, and $\psi \approx -19°$. Assuming that $y' = y(o)$ in Eq.(35), we find that $M = x(o) - y^2(o) \approx .9$ and that the explosive instability occurs at a time $\leq \tau_\infty = 1.4$ linear e-folding times. Since $(\delta/\omega)^2 \ll 1$ is required for these modes, we note that harmonics higher than the second do not satisfy the linear dispersion relation and so need not be included in the calculation.

V. DISCUSSION

We have obtained a general formalism describing the nonlinear evolution of the system (k,ω), $(k_\pm,\omega_\pm)$, $(2k,2\omega)$, $(k_{2\pm},\omega_{2\pm})$ associated with the parametric decay of an intense, coherent electromagnetic wave. This system was reduced to an equivalent two-wave system consisting of the wave fields that determine the first-order electron dynamics, P_k and P_{2k}. The evolutionary equations then assumed a standard form for wave-wave interactions with complex coupling coefficients.

The polar angle , $\psi = arg(c_1c_2)$ determines the essential behavior of the system. In general, the system oscillates in a "trapped region" of ϕ space. If ψ lies in the right-half plane, x and y can increase simultaneously and an explosive instability occurs.

We then applied our results to a modulational mode and found that it can become explosively unstable within a time of the order of a linear e-folding time. These modes, therefore, can grow to an appreciable size of the pump (or perhaps of any other modes that can saturate them) within this time.

APPENDIX - Second Order Perturbed Currents and Densities

First, we calculate $n_k^{(2)}$. According to Eq.(5a),

$$n_k^{(2)} = -\frac{e}{mi}\int \frac{dv}{\omega-\vec{k}\cdot\vec{v}+i\delta}\left[\vec{P}_k^{(2)}\cdot\nabla_v f_o + \vec{P}_{2k}\cdot\nabla_v f_{-k} + \vec{P}_{-k}\cdot\nabla_v f_{2k}\right] \tag{A.1}$$

where ω is real and $\delta\to 0^+$ is the prescription for integrating around the singularity at $\vec{k}\cdot\vec{v}=\omega$. The first term on the right-hand side of Eq.(A.1) is just $-(e/mi)(n_o k^2/\omega_p^2)(P_k^{(2)}/k)\chi(k,\omega)$. The second term can be reduced as follows:

$$\int \frac{dv}{(\omega-\vec{k}\cdot\vec{v}+i\delta)} P_{2k}\cdot\nabla_v f_{-k} = -P_{2k}k\int \frac{dvf_{-k}}{(\omega-\vec{k}\cdot\vec{v}+i\delta)^2}$$

$$= -P_{2k}P_{-k}\frac{ie}{m}\int \frac{dv}{(\omega-\vec{k}\cdot\vec{v}+i\delta)^2}\frac{k\cdot\nabla_v f_o}{(-\omega+\vec{k}\cdot\vec{v}+i\delta)}$$

$$= -P_{2k}P_{-k}\frac{ie}{m}\frac{1}{2i\delta}\int dv\,\vec{k}\cdot\nabla_v f_o\left\{\frac{1}{(\omega-\vec{k}\cdot\vec{v}+i\delta)^2}\right.$$

$$\left.+\frac{1}{2i\delta}\left[\frac{1}{(\omega-\vec{k}\cdot\vec{v}+i\delta)}+\frac{1}{(-\omega+\vec{k}\cdot\vec{v}+i\delta)}\right]\right\}$$

$$= -P_{2k}P_{-k}\frac{ie}{m}\frac{n_o k^2}{\omega_p^2}\frac{1}{2i\delta}\left\{-\frac{\partial}{\partial\omega}\chi(k,\omega+i\delta)+\frac{1}{2i\delta}\left[\chi(k,\omega+i\delta)-\chi(-k,-\omega+i\delta)\right]\right\} \tag{A.2}$$

Since $\chi(k,\omega)$ is analytic in the upper half complex ω plane, χ and its derivatives can be continued analytically down to the real ω axis. We find that

$$\lim_{\delta\to o^+}\int \frac{dv}{(\omega-\vec{k}\cdot\vec{v}+i\delta)} P_{2k}\nabla_v f_{-k} = P_{2k}P_{-k}\frac{ie}{m}\frac{n_o k^2}{\omega_p^2}\frac{1}{2}\frac{\partial^2}{\partial\omega^2}\chi(k,\omega)\quad . \tag{A.3}$$

Similarly,

$$\int \frac{dv}{(\omega-\vec{k}\cdot\vec{v}+i\delta)} P_{-k}\cdot\nabla_v f_{2k} = - P_{-k}P_{2k} \frac{ie}{2m} \int \frac{dv\vec{k}\cdot\nabla_v f_o}{(\omega-\vec{k}\cdot\vec{v}+i\delta)^3}$$

$$\xrightarrow[\delta\to o^+]{} - P_{2k}P_{-k} \frac{ie}{m} \frac{n_o k^2}{\omega_p^2} \frac{1}{4} \frac{\partial^2}{\partial\omega^2} \chi(k,\omega) \tag{A.4}$$

Eq.(A.1) therefore reduces to Eq.(8b). Eq.(8c) is obtained in a similar way.

Next, we examine $j_k^{(2)}$. According to Eq.(5a),

$$\vec{j}_k^{(2)} = \frac{e^2}{mik} \int \frac{dv\,\vec{v}}{(\omega-\vec{k}\cdot\vec{v}+i\delta)} [P_k^{(2)}\,\vec{k}\cdot\nabla_v f_o + P_{2k}\,\vec{k}\cdot\nabla_v f_{-k} + P_{-k}\,\vec{k}\cdot\nabla_v f_{2k}]$$

$$= \frac{e^2\vec{k}}{mi\,k^3} \int \frac{dv[(\vec{k}\cdot\vec{v}-\omega-i\delta)+\omega+i\delta]}{(\omega-\vec{k}\cdot\vec{v}+i\delta)}[P_k^{(2)}\vec{k}\cdot\nabla_v f_0 + P_{2k}\vec{k}\cdot\nabla_v f_{-k} + P_{-k}\vec{k}\cdot\nabla_v f_{2k}]$$

$$= - e\frac{\omega}{k}\, n_k^{(2)} \tag{A.5}$$

The current density $j_{2k}^{(2)}$ is reduced in a similar manner.

The side-band contributions $\vec{j}_\pm^{(2)}$, $\vec{j}_{2\pm}^{(2)}$ are more arduous to obtain; however, since the system is non-relativistic, each integral is easy to evaluate. It is necessary to have expressions for the linear perturbed electron density distributions. The linear Vlasov equations are

$$- i(\omega_o-\vec{k}_o\cdot\vec{v})f_{k_o}^{(1)} - \frac{e}{m}\vec{E}_o\cdot\nabla_v f_o = 0 \;, \tag{A.6}$$

$$- i(\omega_\pm-\vec{k}_\pm\cdot v)f_\pm - \frac{e}{m}(\vec{E}_\pm\cdot\nabla_v f_o + \vec{E}_o\cdot\nabla_v f_k + \vec{E}_k\cdot\nabla_v f_{\pm k_o}) = 0 \;, \tag{A.7}$$

$$- i(\omega-\vec{k}\cdot\vec{v})f_k - \frac{e}{m}\,\vec{E}_k\cdot\nabla_v f_o = 0 \;, \tag{A.8}$$

with similar expressions for f_{2k} , $f_{2\pm}$.

The sideband

$$\vec{j}_{\pm}^{(2)} = \frac{e^2}{mi}\int \frac{dv\,\vec{v}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)}\,[\vec{E}_o\cdot\nabla_v f_k^{(2)} + \vec{E}_k\cdot\nabla_v f_{\pm k_o} + \vec{E}_{2k}\cdot\nabla_v f_{-k\pm k_o}$$

$$+ \vec{E}_{2\pm}\cdot\nabla_v f_{-k}^{(1)} + \vec{E}_{-k}\cdot\nabla_v f_{2\pm}^{(1)} + \vec{E}_{-k\pm k_o}\cdot\nabla_v f_{2k}] \;, \qquad \text{(A.9)}$$

The first term

$$\frac{e^2}{mi}\int \frac{dv\,\vec{v}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)}\,[\vec{E}_o\cdot\nabla_v f_k^{(2)}] =$$

$$-\frac{e^2}{mi}\int dv\left[\frac{\vec{E}_o}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)} + \frac{\vec{k}\cdot\vec{E}_o\vec{v}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)^2}\right] f_k^{(2)} =$$

$$-\frac{e^2}{mi\omega_\pm}\,\vec{E}_o n_k^{(2)} \qquad \text{(A.10)}$$

in lowest order.

In general, $k \lesssim 2k_o$, so that only the first term in Eq. (A.10) is significant. (Modulational modes, in fact, have the property that $\vec{k}\cdot\vec{E}_o = 0$.)

According to Eq. (A.6), $f_{k_o}^{(1)}$ is proportional to E_o . This implies that the second term is, in fact, a contribution to $\vec{j}_{\pm}^{(1)}$ which we find to be down by a factor $[(\omega_p^2)/(\omega_o^2)](1/\chi_e)$. The third term,

$$\frac{e^2}{mi}\int \frac{dv\,\vec{v}\,\vec{E}_{2k}\cdot\nabla_v f_{-k\pm k_o}^{(1)}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)} = -\frac{e^2}{mi}\int dv\left[\frac{\vec{E}_{2k}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v})} + \frac{\vec{v}\,\vec{k}_\pm\cdot\vec{E}_{2k}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v})^2}\right] f_{-k\pm k_0}^{(1)}$$

$$= -\frac{e^2}{mi}\int dv\left[\frac{\vec{E}_{2k}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)} + \frac{\vec{v}\,\vec{k}_\pm\cdot\vec{E}_{2k}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)^2}\right]\frac{ie}{m}\left[\frac{\vec{E}_{-k\pm k_o}\cdot\nabla_v f_o + \vec{E}_o\cdot\nabla_v f_{-k} + \vec{E}_{-k}\cdot\nabla_v f_{\pm k_o}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)}\right]$$

$$= \frac{e^3}{m^2\omega_\pm^3}\left[2\vec{E}_{2k}(\vec{k}_\pm\cdot\vec{E}_{-k\pm k_o}n_o + \vec{k}\cdot\vec{E}_o n_{-k}) + (\vec{k}_\pm\cdot\vec{E}_{2k})(\vec{E}_{-k\pm k_o}n_o + \vec{E}_o n_{-k})\right] \qquad \text{(A.11)}$$

in lowest order.

The fourth term

$$\frac{e^2}{mi}\int dv \frac{\vec{v}\,\vec{E}_{2\pm}\cdot\nabla_v f^{(1)}_{-k}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)} = -\frac{e^2}{mi}\int dv\left[\frac{\vec{E}_{2\pm}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)} + \frac{\vec{v}(\vec{k}_\pm\cdot\vec{E}_{2\pm}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)^2}\right]f^{(1)}_{-k}$$

$$= -\frac{e^2 n^{(1)}_{-k}}{im\omega_\pm}\vec{E}_{2\pm} \qquad \text{(A.12)}$$

in the lowest order. The fifth term

$$\frac{e^2}{mi}\int dv \frac{\vec{v}\,\vec{E}_{-k}\cdot\nabla_v f^{(1)}_{2\pm}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)} = -\frac{e^2}{mi}\int dv\left[\frac{\vec{E}_{-k}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)} + \frac{\vec{v}\,\vec{k}_\pm\cdot\vec{E}_{-k}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)^2}\right]$$

$$\times \frac{ie}{m}\left[\frac{\vec{E}_{2\pm}\cdot\nabla_v f_o + \vec{E}_o\cdot\nabla_v f_{2k} + \vec{E}_{2k}\cdot\nabla_v f_{\pm k_o}}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)}\right]$$

$$= \frac{e^3}{m^2}\int dv\,\vec{E}_{-k}\left[\frac{\vec{k}_\pm(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta) + \vec{k}_{2\pm}(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)^2(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)^2}\right]$$

$$+ \frac{(\vec{k}_\pm\cdot\vec{E}_{-k})\,\underset{\sim}{I}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)^2(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)}$$

$$+ \vec{v}(\vec{k}_\pm\cdot\vec{E}_{-k})\left[\frac{2\vec{k}_\pm(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)+k_{2\pm}(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)^3(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)^2}\right]$$

$$\times [\vec{E}_{2\pm}f_o+\vec{E}_o f_{2k} + \vec{E}_{2k}f_{\pm k_o}]$$

$$= \pm\frac{e^3}{m^2\omega_o^3}[\vec{E}_{-k}(3\vec{k}\pm 2\vec{k}_o)+(\vec{k}_\pm\cdot\vec{E}_{-k})\underset{\sim}{I}]\cdot[\vec{E}_{2\pm}n_o+\vec{E}_o n_{2k}]$$

(A.13)

The last term

$$\frac{e^2}{mi}\int dv \frac{\vec{v}}{(\omega_\pm - \vec{k}_\pm\cdot\vec{v}+i\delta)} \vec{E}_{-k\pm k_o}\cdot\nabla_v f_{2k} \cong -\frac{e^2}{mi}\int dv \frac{\vec{E}_{-k\pm k_o}}{(\omega_\pm - k_\pm\cdot v)} f_{2k}$$

$$= -\frac{e^2 n_{2k}}{mi\omega_\pm} \vec{E}_{-k\pm k_o} , \qquad \text{(A.14)}$$

in the lowest order.

In general, the third and fifth terms are smaller, by at least an order of magnitude, than the other terms and will be neglected. Eq.(A.9) therefore reduces to the following:

$$\vec{j}_\pm^{(2)} = \mp \frac{e^2}{mi\omega_o}\left[n_k^{(2)} \vec{E}_o + n_{-k}^{(1)} \vec{E}_{2\pm} + n_{2k}^{(1)} \vec{E}_{-k\pm k_o}\right] . \qquad \text{(A.15)}$$

Finally, we examine $\vec{j}_{2\pm}^{(2)}$. According to Eq.(5d) ,

$$\vec{j}_{2\pm}^{(2)} = \frac{e^2}{mi}\int dv \frac{\vec{v}}{(\omega_{2\pm} - \vec{k}_{2\pm}\cdot\vec{v}+i\delta)}\left[\vec{E}_o\cdot\vec{\nabla}_v f_{2k} + \vec{E}_{2k}\cdot\nabla_v f_{\pm k_o}\right.$$

$$\left. + \vec{E}_k\cdot\nabla_v f_\pm^{(1)} + \vec{E}_\pm\cdot\nabla_v f_k^{(1)}\right] . \qquad \text{(A.16)}$$

The first term on the right-hand side of Eq.(A.16),

$$\frac{e^2}{mi}\int dv \frac{\vec{v}\,\vec{E}_o\cdot\nabla_v f_{2k}^{(2)}}{(\omega_{2\pm} - \vec{k}_{2\pm}\cdot\vec{v}+i\delta)} = -\frac{e^2}{mi}\int dv\left[\frac{\vec{E}_o}{(\omega_{2\pm} - \vec{k}_{2\pm}\cdot\vec{v}+i\delta)} + \frac{\vec{v}2\vec{k}\cdot\vec{E}_o}{(\omega_{2\pm} - \vec{k}_{2\pm}\cdot\vec{v}+i\delta)^2}\right] f_{2k}^{(2)}$$

$$= -\frac{e^2 n_{2k}^{(2)}}{mi\omega_{2\pm}} \vec{E}_o \qquad \text{(A.17)}$$

in the lowest order. The second term is a small contribution to $\vec{j}^{(1)}_{2\pm}$. The third term

$$\frac{e^2}{mi}\int dv \frac{\vec{v}\,\vec{E}_k\cdot\nabla_v f^{(1)}_\pm}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)} = + \frac{e^3}{(mi)^2}\int dv\left[\frac{\vec{E}_k}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)} + \frac{\vec{v}(\vec{k}_{2\pm}\cdot\vec{E}_k)}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)^2}\right]$$

$$\times \frac{\vec{E}_\pm\cdot\nabla_v f_o + \vec{E}_o\cdot\nabla_v f_k + \vec{E}_k\cdot\nabla_v f_{\pm k_o}}{(\omega_\pm-\vec{k}_\pm\cdot\vec{v}+i\delta)}$$

$$= \mp \frac{e^3}{m^2\omega_o^3}\left[\vec{E}_k(3\vec{k}\pm2\vec{k}_o) + (\vec{k}_{2\pm}\cdot\vec{E}_k)\underset{\sim}{I}\right]\cdot\left[\vec{E}_\pm n_o + \vec{E}_o n_k\right] \tag{A.18}$$

and the fourth term

$$\frac{e^2}{mi}\int dv \frac{\vec{v}\,\vec{E}_\pm\cdot\nabla_v f^{(1)}_k}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)} = - \frac{e^2}{mi}\int dv\left[\frac{\vec{E}_\pm}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)} + \frac{(\vec{k}_{2\pm}\cdot\vec{E}_\pm)\vec{v}}{(\omega_{2\pm}-\vec{k}_{2\pm}\cdot\vec{v}+i\delta)^2}\right] f^{(1)}_k$$

$$= - \frac{e^2 n^{(1)}_k}{mi\omega_{2\pm}}\vec{E}_\pm \quad , \tag{A.19}$$

in lowest order. As in the analysis of $\vec{j}^{(2)}_\pm$, we find that the third term can in general be neglected. Eq.(A.16) therefore takes the form

$$\vec{j}^{(2)}_{2\pm} = \mp \frac{e^2}{mi\omega_o}\left(\vec{E}_o n^{(2)}_{2k} + \vec{E}_\pm n^{(1)}_k\right) \quad . \tag{A.20}$$

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